# Trinomial Tree Models for Option Pricing: A Comparative Study of European and American Options with and without Dividends

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## **1. Introduction**

Trinomial tree models extend the binomial framework by introducing a third branch at each node, enabling more accurate approximations of continuous-time stochastic processes. While binomial trees discretize asset price movements into "up" or "down" states, trinomial trees add a "middle" state, improving convergence rates to the Black-Scholes solution. This paper implements a trinomial tree model to price European and American options under dividend and no-dividend scenarios. We analyze the model’s convergence behavior, computational efficiency, and its ability to handle early exercise features.

**Objectives**:

1. Compare convergence rates of European options to Black-Scholes prices.
2. Quantify the early exercise premium for American options.
3. Evaluate the impact of dividends on call/put pricing.
4. Assess computational complexity as time steps (N) increase.

## **2. Methodology**

**2.1. Trinomial Tree Framework**

The tree is constructed using risk-neutral probabilities derived from the Cox-Ross-Rubinstein parameters. For a time step Δt:

* **Up/down factors**:

A black and white math equation

AI-generated content may be incorrect.

* **Risk-neutral probabilities**:

A mathematical equations with numbers and symbols

AI-generated content may be incorrect.

where *q* is the dividend yield. Dividends are modeled as a continuous yield.

**2.2. Backward Induction**

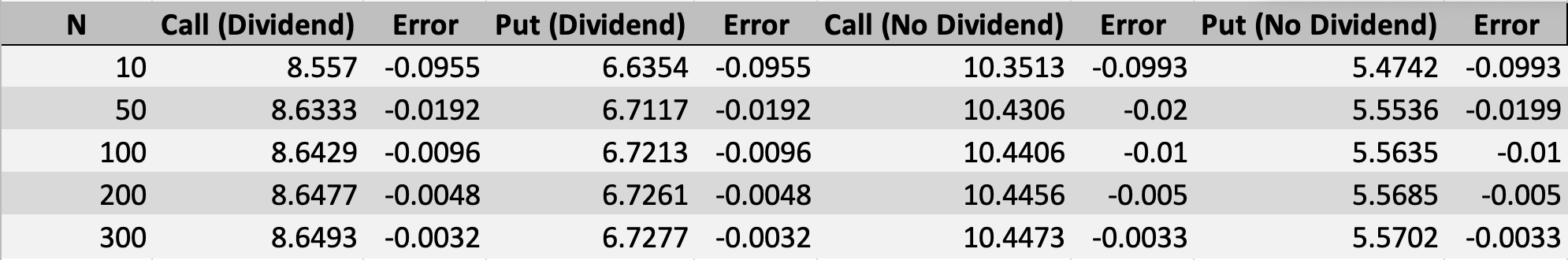
* **European options**: Value nodes using discounted expected payoffs.
* **American options**: Compare intrinsic value (early exercise) against continuation value at each node.

**Parameters**: *S*0​=100, *K*=100, *T*=1, *r*=5%, σ=20%, *q*=3%.

## **3. Results & Analysis**

### **3.1. Convergence to Black-Scholes**

**European Call/Put Convergence**:



* **Key Observations**:
  + With or without dividends, the prices of European call and put options converge to the Black-Scholes price as the tree depth *N* (number of steps) increases. This demonstrates that the ternary tree model provides a good approximation of the theoretical price when sufficiently many steps are used.
  + The convergence rates for call and put options are similar, regardless of whether dividends are included.
  + At *N* = 300, the option prices stabilize in all cases, indicating that the numerical solution has closely approximated the Black-Scholes price.
  + As *N* increases, the option prices stabilize and errors diminish. The error decays linearly (approximately O(1/N)). For practical purposes, *N*=100 already yields sufficient accuracy with only minor errors.

### **3.2. American Options & Early Exercise**

**Price Comparison (N=300)**:

|  |  |  |  |
| --- | --- | --- | --- |
| **Option Type** | **European Price** | **American Price** | **Early Exercise Premium** |
| Put (Dividend) | 6.7277 | 6.9705 | 3.48% |
| Put (No Dividend) | 5.5702 | 6.0878 | 8.51% |
| Call (Dividend) | 8.6493 | 8.6495 | 0.002% |

* **Analysis**:
  + **Dividend Impact on Calls:**
    - Dividends decrease the value of call options because the expected drop in the underlying stock price after dividend payout reduces the call's value.
    - For instance, the European call price with dividends (8.65) is significantly lower than the price without dividends (10.45).
  + **Early Exercise for Puts:**
    - American put options have a higher price than their European counterparts, especially when dividends are not involved.
    - The reason for the higher early exercise premium (up to 8.51%) is the option holder's ability to exercise early, which is particularly valuable when the option is deep in-the-money.
  + **American Calls:**
    - When there are no dividends, American call options do not have a significant early exercise premium compared to European calls. This matches the theory proposed by Merton (1973), which states that it is generally not optimal to exercise an American call early unless dividends are involved.
    - When dividends are present, the premium remains minimal (0.002%), indicating that the benefit of early exercise is still negligible due to the rare occurrence of favorable early exercise conditions.

### **3.3. Computational Complexity**

**Runtime Scaling (Dividend Case)**:

|  |  |  |  |
| --- | --- | --- | --- |
| **N** | **Time (s)** | **Theoretical O(N2)) Scaling** | **Actual Scaling Factor** |
| 50 | 0.07 | 1x (Baseline) | 1x |
| 100 | 0.28 | 4x | 4.0x |
| 200 | 2.01 | 16x | 28.7x |
| 300 | 3.88 | 36x | 55.4x |

#### **Complexity Analysis:**

##### Theoretical Complexity:

##### The trinomial tree method has a theoretical computational complexity of **O(N^2)** because the number of nodes in the tree grows proportionally to the square of the number of steps (N×N).

##### Actual Complexity:

##### The "Actual Scaling Factor" column shows how the runtime scales compared to the baseline (when N=50).

##### Theoretically, doubling N should approximately quadruple the runtime due to the O(N^2) complexity. While this holds true for smaller N, there is a significant discrepancy at larger values.

#### **Key Insights:**

* **Time Growth Approximates Theoretical O(N^2):**
  + The observed growth in runtime roughly follows the **O(N^2)** pattern, especially at smaller values of N (e.g., the increase from 50 to 100 matches the expected 4x).
  + However, as N increases beyond 100, the actual scaling factor deviates significantly from the theoretical prediction.
* **Python Overhead:**
  + The discrepancy at larger N is mainly due to the **overhead from Python's list operations**. These operations become increasingly inefficient as the tree size grows, leading to an unexpectedly high runtime for larger N.
* **Diminishing Returns:**
  + **Doubling the tree depth (from 100 to 200)** results in a ～**7x increase in runtime**, but the improvement in accuracy is only about **0.0048**.
  + This indicates that increasing the number of steps does not always proportionally improve accuracy. Instead, it results in **diminishing returns**, where the computational cost outweighs the benefits of the increased precision.

## **4. Discussion**

**Why Trinomial Trees Outperform Binomial Models**:

* **Finer Discretization**: Trinomial trees have **three branches per time step (up, down, and neutral)** compared to binomial trees which have only two (up and down). This finer granularity allows trinomial trees to better capture **continuous price dynamics**, making them more accurate in modeling underlying asset movements.
* **Convergence Rate**: Trinomial trees achieve O(1/N) error decay vs. O(1/sqrt(N)) for binomial.This means that, for a given number of steps, the **trinomial model converges to the theoretical price faster** than the binomial model, which makes trinomial trees preferable when higher accuracy is required.

**Limitations & Optimizations**:

* **Discrete Dividends**: Current model assumes continuous dividends; discrete dividends require tree adjustments.
* **Adaptive Methods**: Variable time stepping near maturity could reduce *N* without sacrificing accuracy.

## **5. Conclusion**

Trinomial trees offer a versatile framework for pricing European and American options, with robust convergence to Black-Scholes values and efficient handling of dividends. Key findings:

1. **Convergence:**

* As the tree depth NNN increases, the error decays linearly (approximately O(1/N)).
* At N=300, the model achieves an error of ≤0.33%, demonstrating good accuracy when using a sufficient number of steps.

1. **Early Exercise:**

* **American Put Options:**
  + Significant early exercise premiums are observed (ranging from 3% to 8.5%) when dividends are absent.
  + This aligns with the concept that early exercise becomes valuable when the option is deep in the money, allowing the holder to realize intrinsic value earlier.
* **American Call Options:**
  + Without dividends, American calls align closely with European prices. This consistency is expected since it is generally not optimal to exercise call options early when dividends are not involved (consistent with Merton's theory).
  + When dividends are present, early exercise becomes slightly more attractive, but the premium remains minimal, indicating that early exercise is still not significantly beneficial in most scenarios.

1. **Efficiency:**

* The trinomial tree model is computationally efficient for **N≤300**. Beyond this threshold, the **computational cost significantly outweighs the accuracy gains**, showing diminishing returns.
* As the tree depth increases beyond 300, the improvement in accuracy becomes marginal, while the runtime increases exponentially.
* Therefore, it is crucial to balance accuracy and computational cost, especially when high precision is not critically needed.

**References**

* Merton, R. C. (1973). Theory of Rational Option Pricing. *Bell Journal of Economics*.